Fourier transform (general definition)

In general, if G is a locally compact abelian group, then there is unique left translation invariant measure on G (up to scaling), called Haar measure.

- for $G = \mathbb{R}$, Haar measure is the usual Lebesgue measure
- G is discrete, then Haar measure is the counting measure "$\int f \, dg$" $= \sum_{g \in G} f(g)$.
- $G = \mathbb{R}_{>0}^{\times}$, integral given by Haar measure is $\int f(x) \frac{dx}{x}$, since $\frac{dx}{x}$ invariant under multiplication of x by a constant.

Definition: If G locally compact abelian group with Haar measure dg and $f \in L^2(G)$, define $\hat{f} : \hat{G} \to \mathbb{C}$ given by

$$\hat{f}(\chi) = \int_G f(g) \overline{\chi(g)} \, dg .$$

For this course, we'll only be interested in

Theorem 1 (Poisson summation formula).

Let $f \in S(\mathbb{R})$. Then $\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m)$

Proof: Define $F(x) = \sum_{n\in\mathbb{Z}} f(n+x)$.

Then $F \in S(\mathbb{R}/\mathbb{Z})$ (can be verified from definitions)

For $m \in \mathbb{Z}$, $\hat{F}(m) = \int_0^1 F(\theta) e^{-2\pi i m\theta} d\theta$

$$= \int_0^1 \Big(\sum_{n\in\mathbb{Z}} f(n+\theta)\Big) e^{-2\pi i m\theta} d\theta$$

$$= \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(\theta) e^{-2\pi i m(\theta-n)} d\theta$$

$$= \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(\theta) e^{-2\pi i m\theta} d\theta = \hat{f}(m).$$

Hence $F(x) = \sum_{m\in\mathbb{Z}} \hat{F}(m) e^{2\pi i m x} = \sum_{m\in\mathbb{Z}} \hat{f}(m) e^{2\pi i m x}$.

Put $x=0$. $\blacksquare$

Mellin transform

Definition: Let $f : (0,\infty) \to \mathbb{C}$ be such that
- $y^N f(y) \to 0$ as $y \to \infty$, for all $N \in \mathbb{N}$
- There exists $a \in \mathbb{R}$ such that $|y^a f(y)|$ bounded as $y \to 0$.

The Mellin transform of f is the function

$$M(f)(s) = \int_0^\infty f(x)\, x^s \frac{dx}{x}. \quad (*)$$

The integral converges and defines a holomorphic function for $Re(s) > a$.

Remark: $\frac{dx}{x}$ is the invariant measure on $(\mathbb{R}_{>0}, \times)$.

$$\widehat{\mathbb{R}_{>0}} = \{ y \mapsto y^{i\sigma}, \sigma \in \mathbb{R} \} \cong \mathbb{R}.$$

If $\int_0^\infty |f| \frac{dx}{x} < \infty$ (so $f \in L^2(\mathbb{R}_{>0})$)

then the integral $(*)$ converges on $i\mathbb{R}$, and it equals the Fourier tranform on $\mathbb{R}_{>0}$.

We can view Mellin transform as a complex analytic continuation of Fourier transform.

- If $f : \mathbb{R} \to \mathbb{C}$, then $f \circ \log : \mathbb{R}_{>0} \longrightarrow \mathbb{C}$
and $\mathcal{F}(f)(y) = \mathcal{M}(f \circ \log)(-2\pi i y)$
- Similarly, if $g : \mathbb{R}_{>0} \to \mathbb{C}$, then $g \circ \exp : \mathbb{R} \longrightarrow \mathbb{C}$
and $\mathcal{M}(g)(s) = \mathcal{F}(g \circ \exp)\left(-\frac{s}{2\pi i}\right)$

(whenever the transforms are well-defined)

The Mellin transform, is up to scaling by $-2\pi i$, the image of the Fourier transform under the homomorphism $\exp : (\mathbb{R}, +) \to (\mathbb{R}_{>0}, \times)$.

The maps $x \longmapsto x^s$ are quasi-characters of the multiplicative group.

<u>Properties</u>: Let $f : (0,\infty) \to \mathbb{C}$ s.t. $f(x) \ll x^{-a}$ as $x \to 0$
and $f(x) \ll x^{-N}$ as $x \to \infty$, for all $N \in \mathbb{N}$

- Let $u \in \mathbb{R}$ and $v > 0$, define $g(x) = x^u f(x^v)$.
Then $\mathcal{M}(g)(s)$ is defined for $\operatorname{Re}(s) > -u + va$
and it satisfies $\mathcal{M}(g)(s) = \frac{1}{v} \mathcal{M}(f)\left(\frac{s+u}{v}\right)$.
- If f is differentiable, then $\mathcal{M}(f')$ defined for
$\operatorname{Re}(s) > a+1$ and it satisfies $\mathcal{M}(f')(s) = (1-s)\mathcal{M}(f)(s-1)$.

- If $g(x)=f(\alpha x)$, for $\alpha>0$, then $M(g)(s)=\alpha^{-s}M(f)(s)$.
- If $g(x)=f(x^{-1})$, $M(g)$ defined for $Re(s)<-a$ and $M(g)(s)=M(f)(-s)$.
- If $f\in C_c^N((0,\infty))$ (compactly supported, N times differentiable), and $s=\sigma+it$, then
$$M(f)(s)=O_{\sigma,N}((1+|t|)^{-N}).$$

In particular, Mellin transform of a smooth, compactly supported function is entire and rapidly decaying on vertical lines.

Theorem (Mellin Inversion Formula)
Let $f:(0,\infty)\to\mathbb{C}$ smooth, rapidly decaying at ∞ and $f(x)\ll x^{-a}$, as $x\to 0$, for some real number a.

Then
$$f(x)=\frac{1}{2\pi i}\int_{(c)}M(f)(s)\,x^{-s}\,ds,$$
for any real $c>a$

Proof: Replace $f(x)$ by $x^N f(x)$ if necessary, we may assume $a<0$, so can choose $c=0$.

$$\frac{1}{2\pi i}\int_{(0)}M(f)(s)\,x^{-s}\,ds=\frac{1}{2\pi i}\int_{(0)}\mathcal{F}(f\circ\exp)\left(-\frac{s}{2\pi i}\right)x^{-s}\,ds$$

$$= \int_{-\infty}^{\infty} \mathcal{F}(f \circ \exp)(s) \exp(2\pi i s \log x)\, ds$$

$$= f(\exp(\log x)) = f(x).$$

Examples:

- Gamma function

$$\Gamma(s) := \int_0^\infty e^{-y} y^{s-1}\, dy = \mathcal{M}(e^{-y})(s)$$

for $Re(s) > 0$.

$$e^{-x} = \frac{1}{2\pi i} \int_{(1)} \Gamma(s) x^{-s}\, ds.$$

- Perron formula: Let $g(x) = \delta(x^{-1}) = \begin{cases} 1, & 0 < x < 1 \\ 1/2, & x = 1 \\ 0, & x > 1 \end{cases}$

We have that, for $Re(s) > 0$, $M(g)(s) = \frac{1}{s}$.

We saw that, for $c > 0$, $g(x) = \int_{(c)} x^{-s} \frac{1}{s}\, ds$

(central lemma in proof of Perron).

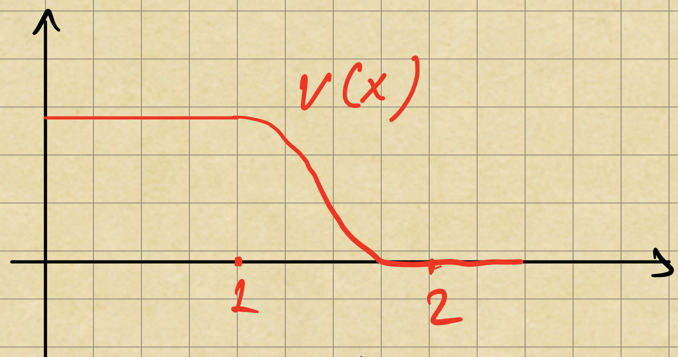
If $\sum f(n) n^{-s}$ absolutely convergent at $s = c$, then

$$\sum_{n \leq x}{}' f(n) = \sum_n f(n) g\left(\frac{n}{x}\right) = \frac{1}{2\pi i} \int_{(c)} L_f(s) x^s \frac{ds}{s}.$$

- Perron formula with smooth cut-off

It is oftentimes useful to use "smooth cut-off" function insted:

Let $V : [0,\infty) \to \mathbb{R}$ smooth compactly supported on $[0,2]$ such that $V(x) = 1$, for $0 \le x \le 1$.



Let $\tilde{V}(s) := \mathcal{M}(V)(s)$.

Then $\tilde{V}(\sigma + it) \ll_N (1+|t|)^{-N}$, $\forall N > 0$, for $-100 < \sigma < 100$

Then $\displaystyle\sum_{n \le x} f(n) V\left(\frac{n}{x}\right) = \frac{1}{2\pi i} \int_{(c)} L_f(s) x^s \tilde{V}(s)\, ds.$

Advantage: $\tilde{V}(s)$ has rapid decay on vertical lines, this will always converge if $L_f(s)$ grows at most polinomially on vertical lines, also easy to bound line integrals if we shift line of integration to apply residue theorem.

- Another example:

$$\sum_n f(n) e^{-\frac{n}{x}} = \frac{1}{2\pi i} \int_{(c)} L_f(s) \Gamma(s) x^s \, ds .$$

Properties of Gamma function

<u>Lemma</u>: For $Re(s) > 0$, we have $\Gamma(s) = \frac{\Gamma(s+1)}{s}$.

<u>Proof</u>: Integration by parts:

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx = \left[\frac{x^s}{s}(-e^{-x})\right]_0^{\infty} - \int_0^\infty \frac{x^s}{s}(-e^{-x})\, dx$$

$$= \frac{\Gamma(s+1)}{s}.$$

<u>Corollary</u>: $\Gamma(s)$ has meromorphic continuation to the entire complex plane, with poles only at the non-positive integers.

<u>Proof</u>: $\frac{\Gamma(s+1)}{s}$ is a meromorphic continuation of $\Gamma(s)$

to $Re(s) > -1$, with a simple pole at $s=0$.

By induction, for any $n \in \mathbb{N}$,

$$\Gamma(s) = \frac{\Gamma(s+n)}{s(s+1)\dots(s+n-1)},$$

and this defines analytic continuation to $Re(s) > -n$ with poles at at $s = 0, -1, \dots -(n-1)$. $\square$

We have $\operatorname{Res}_{s=-n} \Gamma(s) = \frac{(-1)^n}{n!}$ and $\Gamma(n) = (n-1)!$, for $n \in \mathbb{N}$

Can view $\Gamma(s)$ as a generalisation of factorial to complex numbers.

Theorem: For $s \in \mathbb{C} \setminus \mathbb{Z}$, $\Gamma(s) = \lim_{n \to \infty} \frac{n!\, n^s}{s(s+1)\cdots(s+n)}$.

Proof: We see that $\int_0^1 y^{s-1}(1-y)^n \, dy = \frac{n}{s} \int_0^1 y^s (1-y)^{n-1} \, dy$, using integration by parts. Using this recursively,

$$\int_0^1 y^{s-1}(1-y)^n \, dy = \frac{n}{s} \cdot \frac{(n-1)}{s+1} \cdots \frac{1}{s+n-1} \int_0^1 y^{s+n-1} \, dy$$

$$= \frac{n!}{s(s+1)\cdots(s+n)}$$

We do the substitution $y \hookrightarrow \frac{y}{n}$, this becomes

$$\int_0^n y^{s-1}\left(1 - \frac{y}{n}\right)^n dy = \frac{n!\, n^s}{s(s+1)\cdots(s+n)}.$$

Here of course we want to take the limit as $n \to \infty$ and use that $\lim_{n \to \infty} \left(1 - \frac{y}{n}\right)^n = e^{-y}$.

Define $g_n(y) := \begin{cases} y^{s-1}\left(1-\frac{y}{n}\right)^n, & \text{if } y \in [0,n) \\ 0, & \text{otherwise.} \end{cases}$

We have that $\lim_{n\to\infty} g_n(y) = y^{s-1} e^{-y}$.

Since $\left(1-\frac{y}{n}\right)^n \le e^{-y}$, then $|g_n(y)| \le y^{s-1} e^{-y}$.

Therefore, by dominated convergence theorem, we have

for $Re(s)>0$

$$\Gamma(s) = \int_0^\infty \left(\lim_{n\to\infty} g_n(y)\right) dy = \lim_{n\to\infty} \int_0^\infty g_n(y)\, dy$$

$$= \lim_{n\to\infty} \frac{n!\, n^s}{s(s+1)\dots(s+n)}.$$

The conclusion follows from the meromorphic continuation of $\Gamma(s)$. $\square$

Theorem: For all $s \in \mathbb{C}$, $\frac{1}{\Gamma(s)} = s e^{\gamma s} \prod_{n=1}^{\infty} \left(1+\frac{s}{n}\right) e^{-\frac{s}{n}}$,

where γ is the Euler-Mascheroni constant.

Proof: We first note the product converges absolutely for all $s \in \mathbb{C}$. Indeed,

$$\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}} = \prod_n \left(1+\frac{s}{n}\right)\left(1-\frac{s}{n}+O\left(\frac{1}{n^2}\right)\right)$$

$$= \prod_n \left(1 + O\left(\frac{1}{n^2}\right)\right) < \infty.$$

It converges uniformly on compact sets, hence RHS is a holomorphic function on $s \in \mathbb{C}$.

We see that

$$\frac{s(s+1)\dots(s+n)}{n!\, n^s} = s\, n^{-s} \prod_{\ell=1}^{n} \left(1 + \frac{s}{\ell}\right)$$

$$= s \cdot n^{-s} \cdot \prod_{\ell=1}^{n} e^{\frac{s}{\ell}} \cdot \prod_{\ell=1}^{n} \left(1 + \frac{s}{\ell}\right) \cdot e^{-\frac{s}{\ell}}$$

$$= s \exp\left(s\left(\sum_{\ell=1}^{n} \frac{1}{\ell} - \log n\right)\right) \prod_{\ell=1}^{n} \left(1 + \frac{s}{\ell}\right) e^{-\frac{s}{\ell}}$$

Let $n \to \infty$, conclusion follows. $\square$

<u>Theorem</u> (Euler's reflection formula)

For all $s \in \mathbb{C}$, $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$.

<u>Proof</u>: We have that

$$\Gamma(s)\Gamma(1-s) = \lim_{n\to\infty} \left(\frac{n!\, n^s}{s(s+1)\dots(s+n)} \cdot \frac{n!\, n^{1-s}}{(1-s)(2-s)\dots(n+1-s)} \right)$$

$$= \lim_{n\to\infty} \frac{(n!)^2 \cdot n}{s\,(n+1-s)(1-s^2)(2^2-s^2)\dots(n^2-s^2)}$$

$$= \lim_{n\to\infty} \frac{1}{s} \left(1 + \frac{1-s}{n}\right)^{-1} \prod_{\ell=1}^{n} \left(1 - \frac{s^2}{\ell^2}\right)^{-1}.$$

We use that $\frac{\sin(\pi s)}{\pi s} = \prod_{\ell=1}^{\infty}\left(1 - \frac{s^2}{\ell^2}\right)$.

Conclusion follows. $\square$

Remark: This implies $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Corollary: (Non-vanishing of $\Gamma(s)$)

$\Gamma(s)$ has no zeros.

Proof: We know $\sin(\pi s)$ is entire (no poles), hence $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$ has no zeros.

Conclusion follows since poles of $\Gamma(s)$ are at $0 \cup (-\mathbb{N})$, but $\Gamma(n) = (n-1)! \neq 0$, for all $n \in \mathbb{N}$.

$\square$